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Field-based hypotheses on advancing standards for mathematical practice

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ABSTRACT

The Common Core State Standards in Mathematics (CCSSM, 2010) are organized around two types of standards: the standards for mathematical content and standards for mathematical practice. The central goal of this paper is to present cognitive and instructional analyses of standards for mathematical practice through a discussion of field-based activities with inservice secondary mathematics teachers and students. A potential value of the study is that it provides researchers with specific field-based hypotheses on advancing standards for mathematical practice.

1. Introduction

The Common Core State Standards in Mathematics (CCSSM, 2010), following the National Council of Teachers of Mathematics Principles and Standards (NCTM, 2000) and the National Research Council's report, Adding It Up (NRC, 1991), is oriented within a particular perspective of learning, according to which the acquisition of mathematical content evolves hand in hand with the acquisition of mathematical ways of thinking. This perspective is expressed in the composition and narrative of the CCSSM, in that they are comprised of two categories of standards, *standards for mathematical content* and *standards for mathematical practice*, where the latter describe varieties of expertise that students are expected to develop as they engage with mathematical content throughout their school years.

As common-core based curricula are beginning to emerge, there is a need to articulate research questions and hypotheses on the implementation of the CCSSM. The central goal of this paper is to present cognitive and instructional analyses of several of the standards for mathematical practice through a discussion of field-based activities with inservice secondary mathematics teachers and students (hereafter learners or participants).

Our working definition of "field-based hypothesis" is a tentative answer to a research question suggested by observations of learners' mathematical behaviors in authentic classroom settings and is explained by cognitive and instructional analyses oriented within a particular theory of learning, but has not, yet, been proved or disproved by rigorous empirical methodologies. This definition was inspired by the work of Macleod Clark and Hockey (1981), and is consistent with the use of the term "hypothesis" in science, in that it is a proposition not yet verified but set forth to explain certain facts or phenomena in light of established theories.

The rest of the paper is organized around three sections. Section 2 describes the store of data used for the study. The bulk of the data analyzed offered events pertaining to four out of the eight CCSSM practice standards: *model with mathematics, construct viable argument, look for and make use of structure*, and *reason quantitatively*. Accordingly, Section 3 is organized around four subsections discussing cognitive and instructional analyses of these focus standards and the hypotheses they generated. Section 4 concludes with further research questions invoked by these hypotheses; the conceptual framework in which the hypotheses are oriented; and the potential value of the study to practitioners (teachers and curriculum developers).

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2. Source of field-based events

The field-based events discussed in this paper are taken from a series of institutes[1] for inservice secondary mathematics teachers, some in conjunction with teaching experiments with their students, and all were conducted in a large metropolitan region in the western part of the United States. The institutes differed in their scope and duration. The content and level of the institutes varied according to the curricular need and mathematical sophistication of the participants. For example, some institutes targeted the concept of proof across various domains; others focused on particular topics, such as Euclidean geometry, complex numbers, mathematics of consumer finance, theory of systems of linear equations, and functions and their representations. The duration of the institutes also varied considerably. Some were short, lasting one week during the summer or spread in five days throughout an academic year. Others were considerably more extensive: a two-year on-site professional development involving regular class visits followed by conversations with teachers; series of two annual four-week summer institutes, followed by five one-day institutes during each academic year; and series of four annual three-week summer institutes, followed by four one-day institutes during each academic year.

All of our institutes were run as teaching experiments (in the sense of Steffe & Thompson, 2000). The data sources of these teaching experiments also varied considerably. Some institutes included clinical interviews and video recording of small-group discussions and whole-class discussions. In other institutes the data consisted of classroom observations by three individuals (two teacher leaders and a mathematics education researcher) and retrospective notes by the author of this paper, who serves as the teacher-researcher in all the institutes.

The purpose of this description is to highlight the scientific nature of these institutes and the scope of the data they have generated. This paper, however, does not report on this enormous data in any systematic way—that would be beyond the scope of a journal article. Nor does the paper intend to substantiate claims about learners' mathematical behaviors, as is evident in the general nature of the narrative describing the various events. Rather, the paper offers a series of hypotheses on advancing standards for mathematical practice, generated from cognitive and instructional analyses of field-based events.

3. Focus practice standards

In this section, we discuss four out of the eight CCSSM practice standards: *model with mathematics, construct viable argument, look for and make use of structure*, and *reason quantitatively*. These and the rest of the CCSSM practice standards apply to the entire k-12 mathematics curriculum. As such, they are diverse in nature, belonging to different research domains. For example, the above four focus standards belong to the research area of *modeling* (e.g., Lesh & Doerr, 2003), *proof* (e.g., Reid & Knipping, 2010), *structural reasoning* (e.g., Hoch & Dreyfus, 2004), and *quantitative reasoning* (e.g., Thompson, 2011), respectively. Furthermore, each of the practice standards is multidimensional, addressing multiple contexts. For example, the *modeling* standard calls for the development of proficiency to apply "mathematics . . .to solve problems arising in everyday life, society, and the workplace", "use geometry to solve a design problem", "use a function to describe how one quantity of interest depends on another", "making assumptions and approximations to simplify a complicated situation", "identify important quantities in a practical situation", and more. A consequence of this diversity is that this section is unavoidably eclectic, in that the hypotheses, the contexts in which they are discussed, and the analytic tools used to generate them are necessarily heterogeneous. Despite this, all the hypotheses generated are unified by an instructional philosophy underpinning them. The latter point is best explained in the concluding section of the paper after the hypotheses have all been articulated.

This section is organized around the above four focus standards and the data events selected to instantiate them. The "model with mathematics" standard is discussed in the context of events involving functions (Section 3.1); the "construct viable arguments" standard in the context of solving equations (Section 3.2); the "look for and make use of structure" standard in the context of quadratics (Section 3.3); and the "reason quantitatively" standard in the context of algebra word problems (Section 3.4).

3.1. Model with mathematics, with particular reference to functions

The concept of function is central in the CCSSM, and its importance in the learning of mathematics has been well documented by several scholars, most notably Edward Dubinsky (e.g., Ayers, Davis, Dubinsky, & Lewin, 1988; Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Extensive applications of Dubinsky's APOS theory to the development of the concept of function are well reported in the literature of mathematics education. Relevant to the events discussed in this subsection are the *action conception* and *process conception* (corresponding to the initials A and P in APOS). Briefly, action conception of function refers to one's ability to think about the relationship between the dependent variable and independent variable concretely, by seeking to perform the actual computations for a given input to obtain the corresponding output. Process conception of function, on the other hand, refers to one's ability to carry out in thought the input-output process independent of a particular representation and input values.

[1] The term "institute" in this paper refers to an organization set and maintained, typically by a higher-education institution, for a particular period (from one week to several years) for the purpose to advance teachers' knowledge base through a series of periodic professional development activities.

Critical instructional questions concerning the concept of function include: What sort of activities can promote students' understanding of the concept of function at the level of process conception? In what follows, we discuss an activity which typifies the instructional treatment we employed in our institutes to advance the participants' concept of function. The activity revolved around the following problem:

Sliding-line problem: *ABC is a triangle with sides AB* = 6 *m*, *BC* = 8 *m, and AC* = 10 *m. A line l in the plane of the triangle moves along the segment AC at the rate of 1 cm per second. The line starts at A and ends at C, and is always perpendicular to AC.*

a. How long does it take the line to reach the point B?
b. How long does it take the line to bisect the area of the triangle?
c. What is the area of the region that the line sweeps in its movement after (i) 2.25 min, (ii) 3½ min, (iii) 6.75 min?
d. What is the area of the region that the line sweeps at any given moment from the start of its movement?

A central common goal of activities involving problems such as this is to advance the learners' understanding of the concept of function at the level of process conception. The last item (item d) aims at intellectually compelling the learners to reason at this level. Often, learners respond to such items by saying something to the effect that the question cannot be answered since information is missing. "What is the time of the given moment?", asked one of the participants, a behavior akin to an action conception of function. Following the developmental progress depicted in APOS theory, we involved learners in situations similar to items a-c in the above task before item-d-like questions are introduced. The repeated action of determining the output value for a given input value, and vice versa, seemed to have helped learners carry out the respective actions in thought without actually performing them, an essential character of the process conception. An indication for this conceptual development was that the participants, and their own students, were able to describe their planned actions in general terms before performing the actions with particular values. Likewise, we observed learners responding to item-d-like questions by first describing in words the dependency rule between the varying quantities before representing it symbolically.

Often following the completion of the problem some learners point out that all of the preceding items (a–c) can be solved by a simple substitution in the dependency rule constructed in item d, whereby they come to recognize the power of functional representation.

An important feature of item d is that it does not prescribes a solution path by stating something like "Write a formula giving the area *A* as a function of time *t*", as is typically the case in textbooks. In general, a problem that tells the learner the tools to be used or the path to be pursued to obtain a solution (e.g., "use the concept of derivative to determine the . . ." and "express the variable *x* as function of *y* in order to . . .") is dubbed in Harel (2013) *non-holistic.* A holistic problem, on the other hand, is one where the learner is to figure out independently the route to take for problem's solution; the problem statement does not reveal hints or cues as to what routes these might be.

The underlying instructional approach here is that learners are engaged in holistic problems scaffolded in a manner that the learner is intellectually compelled to construct a general representation of input-output values, whereby helping them develop understanding at the level process conception. Through activities revolving around holistic problems of the types discussed here, we observed improved change in the participants' ability to independently construct dependency rules for covarying quantities in various contexts, continuous and discrete. Thus, we posit the following hypothesis:

Hypothesis 1. Engaging learners in holistic problems that intellectually compel them to construct general representations for input–output relations between two varying quantities advances their transition from an action conception of function to a process conception of function.

The CCSSM does not make an explicit distinction between action conception of function and process conception of function. However, statements consistent with the latter conception appear as early as in the eight-grade standards (!); for example, "[eight-grade] Students grasp the concept of a function as a rule that assigns to each input exactly one output, ... understand that functions describe situations where one quantity determines another," (p. 52), and "use functions to model relationships between quantities." (p. 53)

The above hypothesis is companionable to the work of Dubinsky (see, for example, Dubinsky and McDonald, 2001), which has documented the efficacy of programming-based instructional approaches in helping students transition from action conception to process conception. The proposed hypothesis, however, suggests that a different approach—that of applying intellectual need through holistic scaffolding of problematic situations—might be sufficient to achieve similar outcomes.

3.2. Construct viable argument, with particular reference to solving equations

In this section we discuss a field-based event dealing with solution processes for solving linear equations, a topic which often seem mundane, devoid of a need for deep understanding. The activity was designed to problematize this process for our teachers, helping them see that *solving equations is proving!*

The activity started with a simple 2 × 2 linear system:

$$\begin{cases} 2x + y = 7 \\ x - y = 2 \end{cases}$$

Table 1
Faulty solution processes.

John's solution to the equation $\sqrt{2x+1} = -\sqrt{x+2}$	Jill's solution to the equation $2 = \frac{y^2+2y}{y}$
$\sqrt{2x+1} = -\sqrt{x+2}$	$2 = \frac{y^2+2y}{y}$
$\left(\sqrt{2x+1}\right)^2 = \left(-\sqrt{x+2}\right)^2$	$2 = \frac{y^2}{y} + \frac{2y}{y}$
$2x+1 = x+2$	$2 = y+2$
$2x - x = 2 - 1$	$y = 0$
$x = 1$	

The system was then solved by the Elimination-of-Variable method as follows:

$$\begin{cases} 2x+y=7 \\ x-y=2 \end{cases} \to \begin{cases} 2x+y=7 \\ 3x \quad =9 \end{cases} \to \begin{cases} 2x+y=7 \\ x \quad =3 \end{cases} \to \begin{cases} \quad y=1 \\ x \quad =3 \end{cases}$$

On numerous occasions, we asked teachers (and undergraduate students) the following question: "The task given to us is to solve a particular system, but somehow we transition to a sequence of different systems and in the end solve an entirely different system. What justifies this process?" Typically, the teachers either did not understand the question or invoked the term "equivalence" (e.g., "the systems are equivalent"). When they were pressed to explain what this term means, silence followed. On rare occasions, teachers responded that the systems have the same solution set, but were unable to explain why. To further problematize the issue at hand for the teachers, we presented them with a similar sequence of operations applied to a system of linear inequalities:

$$\begin{cases} x+y>2 \\ -3x-y>4 \end{cases}$$

The Elimination-of-Variable method was then applied to obtain:

$$\begin{cases} x+y>2 \\ -3x-y>4 \end{cases} \to \begin{cases} x+y>2 \\ -2x \quad >6 \end{cases} \to \begin{cases} x+y>2 \\ -x \quad >3 \end{cases} \to \begin{cases} \quad y>5 \\ x \quad <-3 \end{cases}$$

The teachers came to realize that something is not right here since solutions to the last system are not necessarily solutions to the original system (e.g., $y=10$ and $x=-4$). This fact puzzled the teachers since they believed that the operations applied were mathematically legitimate, as they were in the case in solving systems of linear equations, and yet they yielded wrong results.

This activity, and others like it, always resulted in great interest and eagerness by the teachers to resolve the quandary. Many lamented that in their own education they have never seen a justification for the Elimination-of-Variable Method; nor did they ever conceived the need to raise questions about its validity; and, furthermore, that they had always assumed that the method is applicable to solving (linear) systems of inequalities.

A follow-up classroom discussion revolved around the core ideas behind the processes applied to the two systems; namely, that while for system of linear equations the operations applied are reversible, for systems of linear inequalities they are not. Follow-up discussions examined the underlying justifications for the standard solution process of equations; and the teachers were able explain the faults in various solution processes, such as the two cases depicted in Table 1.

The teachers indicated that up to this activity they only knew that, as one of them put it, "it is risky to square both sides of equations, but never understood why, nor did it occur to me to ask." They expressed gratification that now they are able to point, with justification, to the faulty steps in the solution process: that in John's solution while $a=b$ implies $a^2=b^2$, $a^2=b^2$ does not imply $a=b$; and in Jill's solution the functions $f(y)=\frac{y^2}{y}$ and $g(y)=y$ are not equal, nor are the functions $h(y)=\frac{2y}{y}$ and $j(y)=2$. Hence, the second and third equations in this solution are not equivalent—the truth of one does not imply the truth of the other.

The teachers in our institutes reported repeatedly that they changed the way they have been teaching procedures (e.g., for solving equations), algorithms (e.g., for operations on whole numbers and fractions), and rules (e.g., for determining the graphical forms of linear and quadratic functions) as a result of the "the revelations [they experienced] in the institute" (one of the teachers' words). They indicated that they have always taken such knowledge for granted, and "never occurred to [them] to even ask why [this knowledge is true]" and that "it is more fun to teach the whys" (participants' statements). Classroom observations of many of these teachers were consistent with these self-reporting statements. Thus, the following hypothesis:

Hypothesis 2. Engaging teachers in activities that problematize the validity of their own knowledge of elementary mathematics changes the way they teach this knowledge—from mere procedures, algorithms, and rules to underlying justifications and proofs.

The practice standard "Construct viable arguments" is very broad. It deals with understanding and using assumptions, definition, and established results; making conjectures and building logical progression to prove them; analyzing cases;

- We wonder if it is possible to write the function $x^2 + 8x - 9$ as a product of the form, $(x + r)(x + t)$.
- We are looking for two numbers, r and , such that $x^2 + 8x - 9 = (x + r)(x + t)$.
- Discuss why we want to do this.
- Multiply out the right-hand side, we get $(x + r)(x + r) = x^2 + (r + t)x + rt$
- Let's now compare the expressions , $x^2 + 8x - 9$ and $x^2 + (r + t)x + rt$:

$$\begin{array}{ccccc} x^2 & + & 8x & - & 9 \\ \downarrow & & \downarrow & & \downarrow \\ x^2 & + & (r+t)x & + & rt \end{array}$$

- The r and t we are looking for are such that their sum is 8 and their product is –9.
- With little guess and check, we find that $r = 9$ and $t = -1$.
- We have successfully factored: $x^2 + 8x - 9 = (x + 9)(x - 1)$
- Thus, our equation has two solutions: $x = -9$, $x = 1$.

Fig. 1. Reducing a quadratic into a product of linear functions.

recognizing and using counter examples; and more. The outcome conjectured in this hypothesis, however limited, is entirely consistent with the CCSSM; namely, proof and justification are to be sought in all contexts and in all grade levels: "One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from." (CCSSM, 2010, p. 6)

The event discussed in this section belongs to a long tradition of the use of puzzles in teaching mathematics (from, for example, Parker (1955) to, for example, Movshovitz-Hadar and Webb (1998)). The central lesson of the event discussed here, however, is not so much about teaching mathematics through fallacies and contradictions; rather, it is about how teachers may shift their instruction of elementary mathematics, from rote procedures to logical underpinning when their own knowledge of these procedures is problematized.

3.3. Look for and make use of structure, with particular reference to quadratics

An essential characteristic of structural reasoning is the understanding that algebraic expressions are manipulated not haphazardly but with the purpose of arriving at a desired form while maintaining certain properties of the expression invariant. In Harel (2008a), this way of thinking is dubbed *algebraic invariance*. Without the algebraic invariance way of thinking, symbol manipulation is largely a mysterious activity for learners—an activity they carry out according to prescribed rules but without a goal in sight. With this way of thinking, on the other hand, symbol manipulation is not a matter of magic tricks but goal-directed operations. The question is how to advance this way of thinking?

We discuss this question in the context of quadratic equations. Typically, completing the square is taught as merely a method for solving quadratic equations (and in some cases it is introduced after the quadratic formula!). While this is indeed one of the functions of the method, our approach was to use the method to also enhance the algebraic invariance way of thinking among learners. To this end, we developed with some of the participants an instructional unit on quadratics along the following series of stages.

The first stage of the unit is a background stage, consisting in activities of solving equations of the forms, $a(x = t)^2 = l$ and $a(x - v)(x - u) = 0$ $(a \neq 0)$, without the use of any formula. The latter form is solved by applying the "zero product principle" and the former by taking the square root of $\frac{l}{a}$ and then solving two simple linear equations. The overall goal is that these two forms become preferred structures—structures into which other unknown forms might be reduced. Accordingly, the next stages involve activities directing students' actions toward manipulating equations of the form $ax^2 + bx + c = 0$ into one of these desirable forms, while maintaining the solution set unchanged.

Stage 2 deals with actions of transforming $ax^2 + bx + c = 0$ into equations of the form $(x - u)(x - v) = 0$. The equations appearing in this stage are all factorable into the latter form where u and v are integers. The factorization process is in the form of an investigation rather than an algorithm, and is based on the previously established fact that two quadratic functions are equal if and only if their coefficients are correspondingly equal. The example in Fig. 1 outlines the progress of such an investigation:

Stage 3 opens with an equation where the previous process fails (because the roots are not integers). The activities in this stage investigate whether a given quadratic $ax^2 + bx + c = 0$ is reducible into an equation of the known form, $a(x = t)^2 = l$. As it can be seen in Fig. 2, the reduction process is analogous to the factorization process in Fig. 1, and it, too, is based on the principle, two quadratics are equal if and only if their corresponding coefficient are equal.

The next three stages consist in activities where the above process (Fig. 2) is repeated for three cases of the coefficients of $ax^2 + bx + c = 0$: (i) $a = 1$, b is a specific constant, and c is a parameter; (ii) $a = 1$, b and c are parameters; (iii) a, b and c are parameters. The transition from Case i to Case ii is also done by reduction to a familiar structure; namely, $ax^2 + bx + c = 0$ is reduced to $x^2 + b'x + c' = 0$, where $b' = \frac{b}{a}$ and $c' = \frac{c}{a}$, whereby arriving at the quadratic formula. Each of these cases is further investigated as to the condition under which the equation has no solution, one solution, or two solutions, leading up to the concept of *discriminant*.

Consider the equation, $x^2 + 18x + 7 = 0$.

- We wonder if it is possible to write the function $x^2 + 18x + 7$ as an expression of the form, $(x + t)^2 - r$.
- Discuss why do we want to do this.
- We are looking for two numbers, t and r, such that the expression $x^2 + 18x + 7$ is equal to the expression $(x + t)^2 - r$.
- Multiply out the last expression we get $(x + t)^2 - r = x^2 + 2tx + t^2 - r$
- Let's now compare the two expressions $x^2 + 18x + 7$ and $x^2 + 2tx + t^2 - r$

$$\begin{array}{ccccc} x^2 & + & 18x & + & 7 \\ \downarrow & & \downarrow & & \downarrow \\ x^2 & + & 2tx & + & (t^2 - r) \end{array}$$

- The t and r we are looking for are such that $2t = 18$ and $t^2 - r = 7$
- No guessing and checking is needed here: $t = 9$ and $r = 74$.
- We now have $x^2 + 18x + 7 = (x + 9)^2 - 74 = 0$
- Hence $(x + 9)^2 = 74$ and so $x + 9 = \pm\sqrt{74}$, or $x = -9 + \sqrt{74}$, $x = -9 - \sqrt{74}$

Fig. 2. Reducing a quadratic into a square of a linear function.

The algebraic invariance way of thinking is applied here through a "nested structure", which is defined as follows: Let ε_1 be a structure familiar to a learner (e.g., $(x+t)^2 = l$). We say that ε_1 is nested in a structure ε_2 newly introduced to the learner (e.g., $ax^2 + bx + c = 0$), if the two structures are, in the eyes of the learner, equivalent with respect to a particular property (e.g., having the same solution set). The goal of instruction is then to explore with students the actions that would transform ε_2 to ε_1 while maintaining certain relevant properties unchanged.

The hypothesis we generated from this kind of tasks with our participants and their students is the following:

Hypothesis 3. Engaging learners in problem-based activities organized in a sequence of "nested structure" advances learners' algebraic invariance way of thinking.

The intellectual gain in acting upon algebraic invariance hypothesis in our institutes was that learners learned that algebraic expressions are re-formed for a reason and accordingly gradually developed a sense of the actions needed in order to reach a desired algebraic form. Furthermore, in the approach proposed in this hypothesis the proof of the quadratic formula is repeatedly used as a tool to solve quadratic equations. Thus, as students apply the algebraic invariance way of thinking, they are also engaged in proving.

Statements in the CCSSM that are consistent with the algebraic invariance way of thinking include "Mathematically proficient students look closely to discern a pattern or structure ... and they can see complicated things, such as some algebraic expressions, as ... being composed of several objects." (p. 8). We point out, however, that while the CCSSM includes illuminating examples of structural reasoning, they do not capture the full scope of this practice, as is discussed in the literature of mathematics education (Harel & Soto, in press).

3.4. Reason quantitatively, with particular reference to algebra word problems

The recognition that quantitative reasoning is an ability that all students can and should develop is not new. Decades ago, Dewey (1933) stressed the need to make the development of reasoning in general, and that about quantities, in particular, a primary goal of school curricula. In recent times, quantitative reasoning continues to be a central goal of mathematics education; for example, it was highlighted as such in statements by the American Mathematical Association of Two-Year Colleges (AMATYC, 1995), the American Mathematical Society (AMS: Howe, 1998), the National Council of Teachers of Mathematics (NCTM, 2000), the Mathematical Association of America (MAA: Sons, 1996), and now in the Common Core State Standards (CCSSM, 2010).

Overwhelming evidence indicates that students across the grades do not learn to reason quantitatively; their mathematical reasoning is mostly quantitative-reasoning free. They manipulate symbols but the manipulation is often divorced from quantitative referents (Stigler et al., 1999; Stigler & Hiebert, 1999) and they focus on what operation the teacher expects them to choose rather than what operations are logically entailed (Sowder, 1988). The lack of attention to quantitative reasoning accounts for this phenomenon, as well as for many other troubling occurrences in students' understanding of key concepts across the grades, especially in algebra (Booth, 1989; Freiman & Lee, 2004; Knuth, Stephens, McNeil, & Alibali, 2006). This literature led us and others (e.g., Thompson, 1993) to conclude that a necessary condition for the development of problem-solving skills is uncompromising attention to quantitative reasoning throughout the curriculum. A common violation of this conclusion is that teachers impose on their students the use of algebraic equations to solve word problems. The following two events illustrate this type of imposition. The events occurred separately in two 10th-grade class, where I was a researcher-observer.

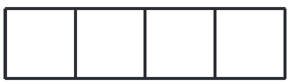
In the first event, the teacher assigned the following in-class problem, where students were asked to work together in small groups.

> Tom and John are roommates. They decided to paint their room. Tom can paint the room in 4 hours and John, a perfectionist, in 8 hours. How long would it take them to paint their room if they work together?

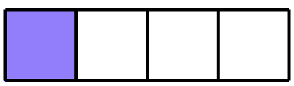
While students work independently in small groups, I was allowed to interact with them freely. I approached one of the groups, observing their interactions. Kate, one of the members in the group, asked me to help them "do the equation". I noticed that on her notebook, there was the expression, $4x + 8x$. The following brief excerpt summarizes our dialogue:

I:	What is x in this expression (pointing to $4x + 8x$)?
Kate:	x? [she and the rest of the group looked at each other; they seem surprised by my question] . . . x mmm, x is . . . the room.
I:	x is the actual room? You want to find x, so you want to find the room?
Kate:	Laughs . . . the rest of group joining in her laughter.

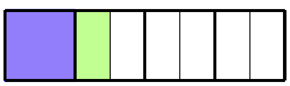
Astonishingly, none of the three students in the group thought about the unknown x as representation of a quantity. To shift their attention to the meaning of the problem statement, I encouraged them to put aside the symbol x and the expression $4x + 8x$ and to act out the problem—to imagine themselves in the position of John and Tom painting their house. My statement to them did not include any clue as to how to go about solving the problem. At this point, I left them on their own, and moved to observe a different group. After about 10 min, Kate raised her hand, gesturing to me to come to her group. On her note there was a rectangle divided into four equal parts, like the following:



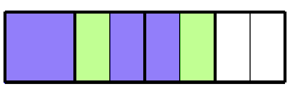
She proceeded to explain that they, the group, figured out that Tom paints 1/4 of the room in 1 h. (She shaded in the first quarter of the rectangle as she uttered these words, as shown in the next figure.)



Then she further divided each of the quarters into two equal parts, and said, again while shading in the adjacent small rectangle to the quarter she already shaded in, that John paints 1/8 of the room in 1 h (next figure).



Then she proceeded by saying something to the effect that it takes 1 h for John and Tom to paint 1/4 +1/8 =3/8 of the room; and it takes another hour for them to paint another 3/8. She accompanied her explanation by continuing shading another 3/8 of her figure (as in the next figure):



"What do we do now?" she asked me. I commended them for their progress, and requested that they continue to work on the problem. A few minutes later, as I was observing another group, Kate waived her hand toward me. "We think we figured it out" she exclaimed. She continued by saying that in 2 h, John and Tom painted 3/4 of the room [marking three shaded-in quarters]. So, it will take them 2/3 of an hour to paint the remaining one quarter. The room will be painted in 2 h and 40 min. She paused for a moment, and then exclaimed: "This can't be right. . . . [pause] . . . There is no *x*".

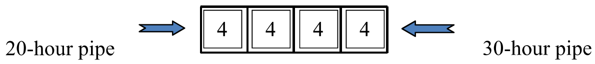
This episode is both remarkable and sad. It is remarkable because the students were able to shift from superficial treatment of the problem to engaging in a conceptual process in a relatively short period of time, what Lesh and Harel (2003) call, *local conceptual development*. It is sad because the students failed to recognize their achievement and deemed their solution incorrect since, according to them, it lacked a form they believed necessary for it to be acceptable by their teacher.

The following episode illustrates further the classroom culture of teachers imposing on their students the use of algebraic equations rather than intellectually necessitating them. The students in a 10^{th}-grade math class were given a test consisting of four problems. A close examination of the students' responses revealed that some students solved the problems correctly but without using equations (explicitly), but the teacher assigned a zero score to these responses. The following is an example of these responses.

One of the problems on the test was:

> A pool is connected to two pipes. One pipe can fill up the pool in 20 hours, and the other in 30 hours. How long will it take the two pipes together to fill up the pool?

Harriet's written solution included a figure like the following:



She accompanied this figure with a few sentences. "The answer is 12 hours. I divided the pool into 5 parts. Three parts [are filled] in 4 + 4 + 4 hour, and two parts in 6 + 6 hours". The teacher's rationale for giving no credit to this solution was something to the effect that Harriet did not follow rules. She did not use algebra and this is an algebra course.

When I interviewed Harriet, she was bitter and not fully cooperative. She lamented that all her answers were correct (they were!), but got no credit for them. She explained her solution, reluctantly by saying something to the effect that it takes the 20-h pipe 4 h to fill up each of the five parts of the pool, and it takes the 30-h pipe 6 h to fill up each of the five parts of the pool. The pool is filled when the two pipes work together for 12 h, because during this time the first pipe fills up 3 parts and the second pipe fills up the remaining two parts. I asked Harriet how she came up with the idea of dividing the pool into 5 equal parts. Her response was laconic: "Wouldn't you?"

This teacher's behavior illustrates the mindset prevalent among many teachers and textbooks. Rather than first developing students' ability to think directly in terms of the quantities involved in the problem using the conceptual tools available to them—thereby developing the habit of building coherent representations for the problem statement and, in turn, advancing their quantitative reasoning—there is a tendency to introduce algebraic symbolism without students experiencing the intellectual need for it.

Teachers in our institutes learn to let their students reason freely with their current conceptual tools and gradually modify the problems to necessitate algebraic treatments. Some of our middle-school teachers reported noticeable changes in their students' ability to represent word problems algebraically after they adopted this approach. To illustrate, consider an event revolved around the following problem:

> Towns A and B are 280 miles apart. At 12:00 PM, a car leaves A toward B, and a truck leaves B toward A. The car drives at 80 m/h and the truck at 60 m/h. When will they meet?

Collectively, the students (eight graders), working in small groups, expressed the following line of reasoning:

> After 1 hour, the car drives 80 miles and truck 60 miles. Together they drive 140 miles. In 2 hour, the car drives 160 miles and the truck 120 miles. Together they drive 280 miles. Therefore, they will meet at 2:00 PM.

The teachers then varied the distance between the two towns through the sequence of numbers, 420, 350, 245, and 309. A few of the students abstracted the structure inherent to the relationship between the problem quantities; namely, that in all cases the time travelled by the two vehicles is the distance between the two towns divided into the sum of the vehicles' speeds. Most of the students, however, attempted to determine the time by guess and check. For example, for the case where the distance between the two towns is 245 miles, these students first concluded that the time it takes until the two vehicles meet must be between 1 and 2 h (already a significant manifestation of quantitative reasoning!), and so they started searching through the values, 1 h and 15 min ($80 \cdot \frac{75}{60} + 60 \cdot \frac{75}{60} \stackrel{?}{=} 245$), 1 h and 30 min ($80 \cdot \frac{90}{60} + 60 \cdot \frac{90}{60} \stackrel{?}{=} 245$), 1 h and 45 min ($80 \cdot \frac{105}{60} + 60 \cdot \frac{105}{60} \stackrel{?}{=} 245$), and find that the last value is the time sought for. This activity of *varying* the time needed was utilized by the teacher to introduce the concept of *variable* (or unknown) and, in turn, to the equation, $80 \cdot x + 60 \cdot x = 140$.

Granted, this is not the only approach to intellectually necessitate the use of algebraic tools for solving word problems. However, whatever approach is used, it is critical to give students ample opportunities to repeatedly reason about the problems with their current arithmetic tools and to gradually help them incorporate new, algebraic tools. This is likely to help students learn to avoid superficial problem-solving approaches, such as that we have seen in the case of Kate and her groupmates.

Events of the kinds we have just discussed led us to the following hypothesis:

Hypothesis 4. Allowing and encouraging learners to apply arithmetic tools available to them to solve word problems help them (a) develop the habit of constructing coherent mental representations for problem statements and, in turn, (b) facilitate the development of algebra-based problem-solving tools.

Part of the instructional approach expressed in this hypothesis is to educate students to build coherent mental representations for the quantities involved in the problem by using their own conceptual tools. This approach is entirely consistent with the CCSSM's characterization of quantitative reasoning as the "habits of creating a coherent representation of the problem at hand". (p. 6)

4. Concluding remarks

In this paper we have discussed various field-based events from the like of which the following hypotheses on advancing certain aspects of the standards for mathematical practice were generated:

1. Engaging learners in holistic problems that intellectually compel them to construct general representations for input-output relations between two varying quantities advances their transition from an action conception of function to a process conception of function.
2. Engaging teachers in activities that problematize the validity of their own knowledge of elementary mathematics changes the way they teach this knowledge—from mere procedures, algorithms, and rules to underlying justifications and proofs.
3. Engaging learners in problem-based activities organized in a sequence of "nested structure" advances learners' algebraic invariance way of thinking.
4. Allowing and encouraging learners to apply arithmetic tools available to them to solve word problems help them (a) develop the habit of constructing coherent mental representations for problem statements and, in turn, (b) facilitate the development of algebra-based problem-solving tools.

4.1. Research questions

These hypotheses are limited to the particular contexts in which they were generated, but they may be viewed as part of broader research questions:

While the concept of function has been broadly investigated, certain questions pertaining to the process conception of function have not, to our knowledge, been explicitly addressed. Examples include: What sort of activities might advance learners' ability not only to construct functions, as is proposed in Hypothesis 1, but also *prove* functional phenomena, such as change of graphical representations due to change of variable (e.g., $y = f(x) \to y = f(x - a)$) or due to change of coordinates (for example, from rectangular coordinates to polar coordinates); relationship between the graph of a function and the graph of its inverse; etc. Proofs of these phenomena require the application of the process conception of function. On some occasions we observed participants attending to such phenomena through input-output considerations, but mostly with limited success. Thus, it seems that engagements in activities of the kind proposed in Hypothesis 1 are not sufficient to deal with proofs of graphical phenomena such as those mentioned above.

Hypothesis 2 is about proving. An extensive review of status studies on students' conceptions of proof by Reid and Knipping (2010) provides a grim picture. This may change if the CCSSM is successfully implemented, due to their focus on understanding through justification. Unfortunately, however, there seems to be a widespread view among teachers that proofs are to be emphasized in college mathematics, whereas in elementary and high-school mathematics, the focus should merely be on computational skills. This discriminating view has been expressed repeatedly by the participants in our institutes, and is consistent with textbooks' approaches to mathematical justifications and proofs (Harel & Wilson, 2011). In particular, we experienced reluctance from some of the teachers in our institutes to engage their students in proving advanced theorems, such as the Rational Root Theorem, the Remainder Theorem, the Chain Theorem, and even geometric constructions. A critical question is: How to help teachers alter this view? What sort of activities bring about a change in teachers' discriminating view about the place and role of proving across the mathematics curriculum, making them understand that attention to proof and justification is not to be preserved for advanced mathematics but should be a central ingredient of the instructional efforts across all grades and mathematics curricula? Activities of the kinds proposed in Hypothesis 2 seem to have mixed results: While more teachers were willing to justify basic rules and procedures as a result of our interventions, only a small number of teachers were agreeable to include proofs in their "advanced" classes.

> In relation to Hypothesis 3, Hoch and Dreyfus (2004) define *structure sense* as the collective ability to see an algebraic expression or sentence as an entity, recognize an algebraic expression or sentence as a previously met structure, divide an entity into sub-structures, recognize mutual connections between structures, recognize which manipulations it is possible to perform, and recognize which manipulations it is useful to perform. (p. 51)

What sort of activities might promote structure sense among learners? Would activities of the kind proposed in Hypotheses 3 further help learners develop structure sense?

Finally, in relation to Hypothesis 4, the process of applying algebraic tools to solve algebra word problems involves the ability to translate a problem statement into an algebraic expression. In this paper, we discussed this ability in the context of elementary mathematical context. This ability, however, is broader, part of to the ability to externalize the exact meaning of concepts and ideas and, if needed, transform them into algebraic expressions. This ability is indispensable in all areas of mathematics, especially in advanced topics. For example, students may understand that to find a least square solution to an inconsistent system $Ax = b$, one needs to replace b by $\hat{b}$, such that $\hat{b}$ is the "closest" to *ColA*, but may experience difficulties reformulating these ideas symbolically into $\hat{b} \in ColA$ and $b - \hat{b} \perp ColA$, and, more crucial, into the equations, $\hat{b} = Ac$ for some vector c and $A^T(b - \hat{b}) = 0$, which are essential to compute $\hat{b}$. The question is what instructional approaches might help learners with this ability? Does the approach proposed in Hypothesis 4 help prepare students for its acquisition?

4.2. The philosophy underlying the proposed hypotheses

Underlying the four hypotheses is the Piagetian view that learners construct knowledge through perturbation. That is, learning is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium (Piaget, 1985; Thompson, 1985; Dubinsky, 1991). An implication for instruction of this view is the *necessity principle:*

> For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to **intellectual need.** (For a discussion on the epistemological, cognitive, and instructional aspects of this principle, see Harel, 2013.)

Intellectual need has to do with disciplinary knowledge born out of a person's current knowledge through engagement in problematic situations conceived as such by her or him. The principle is a theoretical underpinning of the hypotheses discussed in this paper, in that their common essential proposition is that mathematical concepts, skills, and practice should emerge from problem-based activities. While this is not a new idea, the proposed hypotheses spell out specific curricular actions, within an environment where intellectual need is at the center of the instructional effort, that might advance certain aspects of the CCSSM practice standards among students.

Another view underlying the four hypotheses is that mathematical content and mathematical practices are two inseparable categories of knowledge. It is this view that necessitated for us to delve into various content-based analytic tools, such as action conception and process conception (APOS theory), holistic problems versus non-holistic problems, algebraic invariance, nested structure, etc.

This view is fundamental in the CCSSM. Their authors recognize that mathematics is not just subject matter—collections of definitions, theorems, proofs, problems and their solutions, algorithms, etc.—but mathematics also includes *ways of thinking* (i.e., mathematical practices). [2] This recognition implies that instructional objectives throughout each topic in the CCSSM should be formulated in terms of *both* content *and* ways of thinking, not only in terms of the former, as is typically the case in traditional curricula. Thus, understanding and thinking are the main driving forces for the instructional effort of content throughout Common-Core based curricula. The following CCSSM's statement points to the inextricable link between the content standards and the practice standards in reasoning-based curricula:

> The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. . . . [and] . . . effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential points of intersection: between the Standards for Mathematical Content and the Standards for Mathematical Practice. . . . (CCSSM, 2010; p. 8)

4.3. Potential value for practitioners

Evidence exists to indicate that since the publication of the CCSSM, teachers have expressed concerns about the intentions and implications of the new standards. In particular, they voiced concerns about the two types of standards around which the CCSSM are organized: the content standards and mathematical practice standards. Lockwood and Weber (2015) report on a predicament teachers face in their effort to adopt and implement the CCSSM: ". . . teachers have voiced confusion and frustration over the two types of standards [standards for mathematical content and standards for mathematical practices], . . . First they have questions about the distinction between [two types of standards], . . . Second, [they] question how to provide evidence that their students are successfully engaging with appropriate [practice standards]" (p. 461). We hope that this paper will provide practitioners, both teachers and curriculum developers, an improved understanding of the role of the practice standards in mathematics instruction put forth by the CCSSM.

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[2] For philosophical and cognitive debates of this claim, see Harel (2008a).

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